General Linear Method for Solving Various Types of Differential Equations

Faranak Rabiei

Institute For Mathematical Research
Universiti Putra Malaysia

SAWONA 2018, 3-4 April 2018

Outline

The idea of General Linear Methods came from the generalizations of traditional methods and it appears in Butcher (1985). He proposed that the generalization of both multivalue multistep method and multistage Runge-Kutta method in a natural way leads to GLM. A comprehensive study on GLM later published in Butcher (2006, 2008).

Consider the first order ordinary differential equation:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

The general formula of General Linear Method:

$$Y_i = \sum_{j=1}^{s} a_{ij} h F_j + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \qquad i = 1, 2, \ldots, s,$$

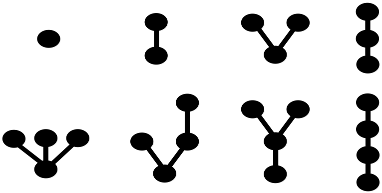
$$y_i^{[n]} = \sum_{j=1}^{s} b_{ij} h F_j + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \qquad i = 1, 2, \ldots, r.$$

Coefficients of GLM could be represented in Butcher's Table:

$$\begin{array}{c|c} A_{s\times s} & U_{s\times r} \\ \hline B_{r\times s} & V_{r\times r} \end{array}$$

The GLM method suggested here is an order four method which are associated with a set of 8 "rooted trees". This trees are denoted as t:

The GLM method suggested here is an order four method which are associated with a set of 8 "rooted trees". This trees are denoted as t:



- To obtain values of coefficients for GLM, we need to solve several order conditions.

- To obtain values of coefficients for GLM, we need to solve several order conditions.
- Derivation of order conditions is based on two equations given:

$$\boxed{\eta(t) = A(\eta D)(t) + U\xi(t)}$$

$$\boxed{E\xi(t) = B(\eta D)(t) + V\xi(t)}$$

where ξ represents the input approximation computed by a starting method, the stage values are denoted by η and the stage derivatives are represented by ηD.

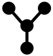
Matrix representation of coefficients for GLM with $s = 3$, $r = 2$:

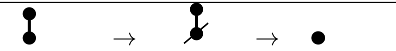
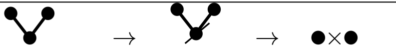
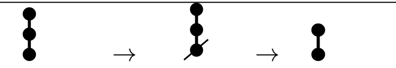
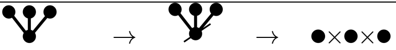
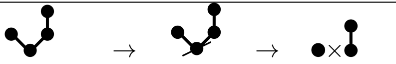
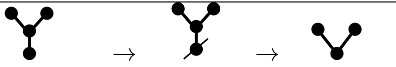
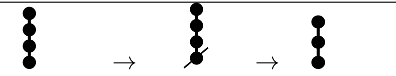
$$
\left[\begin{array}{c|c}
A = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} & U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \\
\hline
B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} & V = \begin{bmatrix} 1 & v_{12} \\ v_{21} & 0 \end{bmatrix}
\end{array}\right]
$$

General Linear Method

Values of $\xi(t)$ and $E\xi(t)$ for $t \in T^{\#}$

Before applying Theorem 2.1, we outset the values of $\xi(t) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ and $E\xi(t) = \begin{bmatrix} E\xi_1 \\ E\xi_2 \end{bmatrix}$ for $T^{\#} = \{t_0, t_1, \ldots, t_8\}$:

t	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
tree	$\emptyset$								
ξ_1	1	0	0	0	0	0	0	0	0
ξ_2	1	-1	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}
$E\xi_1$	1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$
$E\xi_2$	1	-1	0	0	0	0	0	0	0

t	Operation	
t_1		$\Rightarrow (\eta D)(t_1) = \eta(t_0)$
t_2		$\Rightarrow (\eta D)(t_2) = \eta(t_1)$
t_3		$\Rightarrow (\eta D)(t_3) = \eta(t_1^2)$
t_4		$\Rightarrow (\eta D)(t_4) = \eta(t_2)$
t_5		$\Rightarrow (\eta D)(t_5) = \eta(t_1^3)$
t_6		$\Rightarrow (\eta D)(t_6) = \eta(t_1 t_2)$
t_7		$\Rightarrow (\eta D)(t_7) = \eta(t_3)$
t_8		$\Rightarrow (\eta D)(t_8) = \eta(t_4)$

General Linear Method

Calculation of $\eta(t)$

$$
\begin{aligned}
\eta(\emptyset) &= A(\eta D)(\emptyset) + U\xi(\emptyset) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} u_{11} + u_{12} \\ u_{21} + u_{22} \\ u_{31} + u_{32} \end{bmatrix}. \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\eta(t_1) &= A(\eta D)(t_1) + U\xi(t_1) \\
&= A\eta(t_0) + U\xi(t_1) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} + u_{12} \\ u_{21} + u_{22} \\ u_{31} + u_{32} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -u_{12} \\ a_{21}(u_{11} + u_{12}) - u22 \\ a_{31}(u_{11} + u_{12}) + a_{32}(u_{21} + u_{22}) - u_{32} \end{bmatrix}. \qquad (2)
\end{aligned}
$$

$$\begin{aligned}
(E\xi)(\emptyset) &= B(\eta D)(\emptyset) + V\xi(\emptyset) \\
\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & v_{12} \\ v_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 + v_{12} \\ v_{21} \end{bmatrix}.
\end{aligned} \tag{3}$$

$$\begin{aligned}
(E\xi)(t_1) &= B(\eta D)(t_1) + V\xi(t_1) \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= B\eta(t_0) + V\xi(t_1) \\
&= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} + \begin{bmatrix} 1 & v_{12} \\ v_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} b_{11}(u_{11} + u_{12}) + b_{12}(u_{21} + u_{22}) + b_{13}(u_{31} + u_{32}) - v_{12} \\ b_{21}(u_{11} + u_{12}) + b_{22}(u_{21} + u_{22}) + b_{23}(u_{31} + u_{32}) \end{bmatrix}.
\end{aligned} \tag{4}$$

General Linear Method

Order conditions of GLM

Order conditions for third order GLM wih $s = 3, r = 2$

No	Order conditions
1	$b_{11}(u_{11}+u_{12})+b_{12}(u_{21}+u_{22})+b_{13}(u_{31}+u_{32})-v_{12}=1$
2	$b_{21}(u_{11}+u_{12})+b_{22}(u_{21}+u_{22})+b_{23}(u_{31}+u_{32})=0$
3	$-b_{11}u_{12}+b_{12}(a_{21}(u_{11}+u_{12})-u_{22})+b_{13}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})-u_{32})$ $+v_{12}\xi_{22}=\frac{1}{2}$
4	$-b_{21}u_{12}+b_{22}(a_{21}(u_{11}+u_{12})-u_{22})+b_{23}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})-u_{32})=0$
5	$b_{11}u_{12}^2+b_{12}(a_{21}(u_{11}+u_{12})-u_{22})^2+b_{13}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})-u_{32})^2$ $+v_{12}\xi_{23}=\frac{1}{3}$
6	$b_{21}u_{12}^2+b_{22}(a_{21}(u_{11}+u_{12})-u_{22})^2+b_{23}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})-u_{32})^2=0$
7	$b_{11}u_{12}\xi_{22}+b_{12}(\xi_{22}u_{22}-a_{21}u_{12})+b_{13}(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})+u_{32}\xi_{22})$ $+v_{12}\xi_{24}=\frac{1}{6}$
8	$b_{21}u_{12}\xi_{22}+b_{22}(\xi_{22}u_{22}-a_{21}u_{12})+b_{23}(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})$ $+u_{32}\xi_{22})=0$
9	$-b_{11}u_{12}^3+b_{12}(a_{21}(u_{11}+u_{12})-u_{22})^3+b_{13}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})-u_{32})^3$ $+v_{12}\xi_{25}\frac{1}{4}=\frac{1}{4}$
10	$-b_{21}u_{12}^3+b_{22}(a_{21}(u_{11}+u_{12})-u_{22})^3+b_{23}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}+u_{22})$ $-u_{32})^3=0$
11	$-b_{11}u_{12}^2\xi_{22}+b_{12}(a_{21}(u_{11}+u_{12})-u_{22})(\xi_{22}u_{22}-a_{21}u_{12})+b_{13}(a_{31}(u_{11}+u_{12})+a_{32}(u_{21}$ $+u_{22})-u_{32})(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})+u_{32}\xi_{22})+v_{12}\xi_{26}=\frac{1}{8}$
12	$-b_{21}u_{12}^2\xi_{22}+b_{22}(a_{21}(u_{11}+u_{12})-u_{22})(\xi_{22}u_{22}-a_{21}u_{12})+b_{23}(a_{31}(u_{11}+u_{12})$ $+a_{32}(u_{21}+u_{22})-u_{32})(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})+u_{32}\xi_{22})=0$
13	$b_{11}u_{12}\xi_{23}+b_{12}(a_{21}u_{12}^2+\xi_{23}u_{22})+b_{13}(a_{31}u_{12}^2+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^2+u_{32}\xi_{23})$ $+v_{12}\xi_{27}=\frac{1}{12}$
14	$b_{21}u_{12}\xi_{23}+b_{22}(a_{21}u_{12}^2+\xi_{23}u_{22})+b_{23}(a_{31}u_{12}^2+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^2$ $+u_{32}\xi_{23})=0$
15	$b_{11}u_{12}\xi_{24}+b_{12}(\xi_{22}a_{21}u_{12}+\xi_{24}u_{22})+b_{13}(a_{31}u_{12}\xi_{22}+a_{32}(\xi_{22}u_{22}-a_{21}u_{12})+u_{32}\xi_{24})$ $+v_{12}\xi_{28}=\frac{1}{24}$
16	$b_{21}u_{12}\xi_{24}+b_{22}(\xi_{22}a_{21}u_{12}+\xi_{24}u_{22})+b_{23}(a_{31}u_{12}\xi_{22}+a_{32}(\xi_{22}u_{22}-a_{21}u_{12})$ $+u_{32}\xi_{24})=0$

Order conditions for fourth order GLM wih $s = 3, r = 2$

No	Order conditions
1	$b_{11}u_{12}{}^4 + b_{12}\left(a_{21}\left(u_{11}+u_{12}\right)-u_{22}\right)^4 + b_{13}\left(a_{31}\left(u_{11}+u_{12}\right)+a_{32}\left(u_{21}+u_{22}\right)-u_{32}\right)^4 + v_{12}\xi_{29} = \frac{5}{24}$
2	$b_{21}u_{12}{}^4 + b_{22}\left(a_{21}\left(u_{11}+u_{12}\right)-u_{22}\right)^4 + b_{23}\left(a_{31}\left(u_{11}+u_{12}\right)+a_{32}\left(u_{21}+u_{22}\right)-u_{32}\right)^4=0$
3	$b_{11}u_{12}{}^3\xi_{22} + b_{12}(a_{21}(u_{11}+u_{12})-u_{22})^2(-a_{21}u_{12}+u_{22}\xi_{22}) + b_{13}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})^2(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})+u_{32}\xi_{22}) + v_{12}\xi_{210} = \frac{5}{48}$
4	$b_{21}u_{12}{}^3\xi_{22} + b_{22}(a_{21}(u_{11}+u_{12})-u_{22})^2(-a_{21}u_{12}+u_{22}\xi_{22}) + b_{23}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})^2(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})+u_{32}\xi_{22}) = 0$
5	$-b_{11}u_{12}{}^2\xi_{23} + b_{12}(a_{21}(u_{11}+u_{12})-u_{22})(a_{21}u_{12}{}^2+u_{22}\xi_{23}) + b_{13}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})(a_{31}u_{12}{}^2+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^2+u_{32}\xi_{23}) + v_{12}\xi_{211} = \frac{1}{9}$
6	$-b_{21}u_{12}{}^2\xi_{23} + b_{22}(a_{21}(u_{11}+u_{12})-u_{22})(a_{21}u_{12}{}^2+u_{22}\xi_{23}) + b_{23}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})(a_{31}u_{12}{}^2+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^2+u_{32}\xi_{23}) = 0$
7	$-b_{11}u_{12}{}^2\xi_{24} + b_{12}(a_{21}(u_{11}+u_{12})-u_{22})(a_{21}u_{12}\xi_{22}+u_{22}\xi_{24}) + b_{13}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})(a_{31}u_{12}\xi_{22}+a_{32}(-a_{21}u_{12}+u_{22}\xi_{22})+u_{32}\xi_{24}) + v_{12}\xi_{212} = \frac{1}{18}$
8	$-b_{21}u_{12}{}^2\xi_{24} + b_{22}(a_{21}(u_{11}+u_{12})-u_{22})(a_{21}u_{12}\xi_{22}+u_{22}\xi_{24}) + b_{23}(a_{31}(u_{11}+u_{12}) + a_{32}(u_{21}+u_{22})-u_{32})(a_{31}u_{12}\xi_{22}+a_{32}(-a_{21}u_{12}+u_{22}\xi_{22})+u_{32}\xi_{24}) = 0$
9	$b_{11}u_{12}{}^2\xi_{22}{}^2 + b_{12}(-a_{21}u_{12}+u_{22}\xi_{22})^2 + b_{13}(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22}) + u_{32}\xi_{22})^2 + v_{12}\xi_{213} = \frac{5}{96}$
10	$b_{21}u_{12}{}^2\xi_{22}{}^2 + b_{22}(-a_{21}u_{12}+u_{22}\xi_{22})^2 + b_{23}(-a_{31}u_{12}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22}) + u_{32}\xi_{22})^2 = 0$
11	$b_{11}u_{12}\xi_{25} + b_{12}(-a_{21}u_{12}{}^3+u_{22}\xi_{25}) + b_{13}(-a_{31}u_{12}{}^3+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^3 + u_{32}\xi_{25}) + v_{12}\xi_{214} = \frac{1}{24}$
12	$b_{21}u_{12}\xi_{25} + b_{22}(-a_{21}u_{12}{}^3+u_{22}\xi_{25}) + b_{23}(-a_{31}u_{12}{}^3+a_{32}(a_{21}(u_{11}+u_{12})-u_{22})^3 + u_{32}\xi_{25}) = 0$
13	$b_{11}u_{12}\xi_{26} + b_{12}(-a_{21}u_{12}{}^2\xi_{22}+u_{22}\xi_{26}) + b_{13}(-a_{31}u_{12}{}^2\xi_{22}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22}) (-a_{21}u_{12}+u_{22}\xi_{22})+u_{32}\xi_{26}) + v_{12}\xi_{215} = \frac{1}{48}$
14	$b_{21}u_{12}\xi_{26} + b_{22}(-a_{21}u_{12}{}^2\xi_{22}+u_{22}\xi_{26}) + b_{23}(-a_{31}u_{12}{}^2\xi_{22}+a_{32}(a_{21}(u_{11}+u_{12})-u_{22}) (-a_{21}u_{12}+u_{22}\xi_{22})+u_{32}\xi_{26}) = 0$
15	$b_{11}u_{12}\xi_{27} + b_{12}(a_{21}u_{12}\xi_{23}+u_{22}\xi_{27}) + b_{13}(a_{31}u_{12}\xi_{23}+a_{32}(a_{21}u_{12}{}^2+u_{22}\xi_{23})+u_{32}\xi_{27}) + v_{12}\xi_{216} = -\frac{1}{36}$
16	$b_{21}u_{12}\xi_{27} + b_{22}(a_{21}u_{12}\xi_{23}+u_{22}\xi_{27}) + b_{23}(a_{31}u_{12}\xi_{23}+a_{32}(a_{21}u_{12}{}^2+u_{22}\xi_{23})+u_{32}\xi_{27}) = 0$
17	$b_{11}u_{12}\xi_{28} + b_{12}(a_{21}u_{12}\xi_{24}+u_{22}\xi_{28}) + b_{13}(a_{31}u_{12}\xi_{24}+a_{32}(a_{21}u_{12}\xi_{22}+u_{22}\xi_{24})+u_{32}\xi_{28})$

- Using the order conditions given in before, for $t_0 \ldots t_3$ and minimizing the error norm of GLM of order four (for trees $t_4 \ldots t_8$) the GLM of order three is derived. In this way, we set the values $b_{11} = \frac{1}{6}, b_{12} = \frac{2}{3}, b_{13} = \frac{1}{6}, u_{11} = 1, u_{12} = 0, u_{21} = \frac{7}{9}, u_{12} = \frac{2}{9}$ and substitute the values of ξ and $E\xi$.

General Linear Method

Calculation of order for GLM

- Using the order conditions given in before, for $t_0 \dots t_3$ and minimizing the error norm of GLM of order four (for trees $t_4 \dots t_8$) the GLM of order three is derived. In this way, we set the values $b_{11} = \frac{1}{6}, b_{12} = \frac{2}{3}, b_{13} = \frac{1}{6}, u_{11} = 1, u_{12} = 0, u_{21} = \frac{7}{9}, u_{12} = \frac{2}{9}$ and substitute the values of ξ and $E\xi$.
- The coefficients of three stages third order GLM method is shown in Table 5. The calculation of various quantities assigned to obtain the method can be found in Table 1.

$c_1 = 0$				$u_{11} = 1$	$u_{12} = 0$
$c_2 = \frac{1}{2}$	$a_{21} = \frac{13}{18}$			$u_{21} = \frac{7}{9}$	$u_{22} = \frac{2}{9}$
$c_3 = 1$	$a_{31} = -\frac{17}{9}$	$a_{32} = 2$		$u_{21} = \frac{17}{9}$	$u_{22} = -\frac{8}{9}$
	$b_{11} = \frac{1}{6}$	$b_{12} = \frac{2}{3}$	$b_{13} = \frac{1}{6}$	$v_{11} = 1$	$v_{12} = 0$
	$b_{21} = 0$	$b_{22} = 0$	$b_{23} = 0$	$v_{21} = 1$	$v_{22} = 0$

Table 1: Coefficients set of third order GLM

Calculation of order for GLM

t	ξ_1	ξ_2	η_1	$\eta_1 D$	η_2	$\eta_2 D$	η_3	$\eta_3 D$	$E\xi_1$	$E\xi_2$
t_1	0	-1	0	1	$\frac{1}{2}$	1	1	1	1	0
t_2	0	$\frac{9}{16}$	0	0	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0
t_3	0	$\frac{-1}{3}$	0	0	$\frac{-2}{27}$	$\frac{1}{4}$	$\frac{43}{54}$	1	$\frac{1}{3}$	0
t_4	0	$\frac{-1}{6}$	0	0	$\frac{-1}{27}$	$\frac{1}{8}$	$\frac{43}{108}$	$\frac{1}{2}$	$\frac{1}{6}$	0
t_5	0	$\frac{1}{4}$	0	0	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{1}{36}$	1	$\frac{1}{4}$	0
t_6	0	$\frac{1}{8}$	0	0	$\frac{1}{36}$	$\frac{1}{16}$	$\frac{1}{72}$	$\frac{1}{2}$	$\frac{1}{8}$	0
t_7	0	$\frac{1}{12}$	0	0	$\frac{1}{54}$	$\frac{-2}{27}$	$\frac{-2}{9}$	$\frac{43}{54}$	$\frac{1}{12}$	0
t_8	0	$\frac{1}{24}$	0	0	$\frac{1}{108}$	$\frac{-1}{27}$	$\frac{-1}{9}$	$\frac{43}{108}$	$\frac{1}{24}$	0

By solving the order conditions, we obtain

Table 2a. Coefficients of GLM Set 1

$c_1 = 0$				$u_{11} = 1$	$u_{12} = 0$
$c_2 = \frac{1}{2}$	$a_{21} = \frac{13}{18}$			$u_{21} = \frac{7}{9}$	$u_{22} = \frac{2}{9}$
$c_3 = 1$	$a_{31} = \frac{-17}{9}$	$a_{32} = 2$		$u_{31} = \frac{17}{9}$	$u_{32} = \frac{-8}{9}$
	$b_{11} = \frac{1}{6}$	$b_{12} = \frac{2}{3}$	$b_{13} = \frac{1}{6}$	$v_{11} = 1$	$v_{12} = 0$
	$b_{21} = 0$	$b_{22} = 0$	$b_{23} = 0$	$v_{21} = 1$	$v_{22} = 0$

By solving the order conditions, we obtain

Table 2b. Coefficients of GLM Set 2

$c_1 = 0$				$u_{11} = 1$	$u_{12} = 0$
$c_2 = \frac{1}{2}$	$a_{21} = \frac{5}{6}$			$u_{21} = \frac{2}{3}$	$u_{22} = \frac{1}{3}$
$c_3 = 1$	$a_{31} = \frac{-7}{3}$	$a_{32} = 2$		$u_{31} = \frac{7}{3}$	$u_{32} = \frac{-4}{3}$
	$b_{11} = \frac{1}{6}$	$b_{12} = \frac{2}{3}$	$b_{13} = \frac{1}{6}$	$v_{11} = 1$	$v_{12} = 0$
	$b_{21} = 0$	$b_{22} = 0$	$b_{23} = 0$	$v_{21} = 1$	$v_{22} = 0$

By solving the order conditions, we obtain

Table 2c. Coefficients of GLM Set 3

$c_1 = 0$				$u_{11} = 1$	$u_{12} = 0$
$c_2 = \frac{1}{2}$	$a_{21} = \frac{2}{3}$			$u_{21} = \frac{5}{6}$	$u_{22} = \frac{1}{6}$
$c_3 = 1$	$a_{31} = \frac{-5}{3}$	$a_{32} = 2$		$u_{31} = \frac{5}{3}$	$u_{32} = \frac{-2}{3}$
	$b_{11} = \frac{1}{6}$	$b_{12} = \frac{2}{3}$	$b_{13} = \frac{1}{6}$	$v_{11} = 1$	$v_{12} = 0$
	$b_{21} = 0$	$b_{22} = 0$	$b_{23} = 0$	$v_{21} = 1$	$v_{22} = 0$

General Linear Method

Error Norm

The principal error norm is expressed as:

$$||\tau^{p+1}||_2 = \sqrt{\sum_{j=1}^{n_{p+1}} \left(\tau_j^{(p+1)}\right)^2} \tag{5}$$

The equations of order condition for fourth order GLM which are not satisfied are numbered (5), (7), (15) and (17) given as:

$$\tau_5^4 : -b_{11}{u_{12}}^2\xi_{23} + b_{12}(a_{21}(u_{11}+u_{12}) - u_{22})(a_{21}{u_{12}}^2 + \xi_{23}u_{22}) + b_{13}(a_{31}(u_{11}+u_{12}) \\ + a_{32}(u_{21}+u_{22}) - u_{32})(a_{31}{u_{12}}^2 + a_{32}(a_{21}(u_{11}+u_{12}) - u_{22})^2 + u_{32}\xi_{23}) + v_{12}\xi_{211} = \tag{6}$$

$$\tau_7^4 : -b_{11}{u_{12}}^2\xi_{24} + b_{12}(a_{21}(u_{11}+u_{12}) - u_{22})(\xi_{22}a_{21}u_{12} + E_{24}u_{22}) + b_{13}(a_{31}(u_{11}+u_{12}) \\ + a_{32}(u_{21}+u_{22}) - u_{32})(a_{31}u_{12}\xi_{22} + a_{32}(\xi_{22}u_{22} - a_{21}u_{12}) + u_{32}\xi_{24}) + v_{12}\xi_{212} = \frac{1}{18} \tag{7}$$

$$\tau_{15}^{4} : b_{11}u_{12}\xi_{27} + b_{12}(\xi_{23}a_{21}u_{12} + \xi_{27}u_{22}) + b_{13}(a_{31}u_{12}\xi_{23} + a_{32}(a_{21}{u_{12}}^{2} + \xi_{23}u_{22}) + u_{32}\xi_{2}$$
$$+ v_{12}\xi_{216} = -\frac{1}{36} \tag{8}$$

$$\tau_{17}^{4} : b_{11}u_{12}\xi_{28} + b_{12}(\xi_{24}a_{21}u_{12} + \xi_{28}u_{22}) + b_{13}(a_{31}u_{12}\xi_{24} + a_{32}(\xi_{22}a_{21}u_{12} + \xi_{24}u_{22}) + u_{3}$$
$$+ v_{12}\xi_{217} = -\frac{1}{72} \tag{9}$$

Substituting coefficients Set 1 and assuming $\xi_{23} = -\frac{1}{3}$, $\xi_{24} = -\frac{1}{6}$ into equations 6 - 9, the principal error norm for Set 1 using equation 5 is given as

$$\begin{aligned} ||\tau^{4}||_{2} &= \sqrt{\left({\tau_{5}^{4}}^{2} + {\tau_{7}^{4}}^{2} + {\tau_{15}^{4}}^{2} + {\tau_{17}^{4}}^{2}\right)} \\ &= 0.004880058119. \end{aligned} \tag{10}$$

Substituting coefficients Set 2 and assuming $\xi_{23} = -\frac{1}{3}$, $\xi_{24} = -\frac{1}{6}$ into equations 6 - 9, the principal error norm for Set 2 using equation 5 is given as

$$||\tau^4||_2 = \sqrt{\left({\tau_5^4}^2 + {\tau_7^4}^2 + {\tau_{15}^4}^2 + {\tau_{17}^4}^2\right)} = 0.01464017435. \quad (11)$$

Substituting coefficients Set 3 and assuming $\xi_{23} = -\frac{1}{3}$, $\xi_{24} = -\frac{1}{6}$ into equations 6 - 9, the principal error norm for Set 3 using equation 5 is given as

$$||\tau^4||_2 = \sqrt{\left({\tau_5^4}^2 + {\tau_7^4}^2 + {\tau_{15}^4}^2 + {\tau_{17}^4}^2\right)} = 0.01464017435. \quad (12)$$

By comparing the obtained values of error norm in 10 - 12, Set 1 of coefficients of GLM performed smaller error norm compared with two other sets.

h	:	step size
MAXE	:	maximum error
TS	:	total steps taken
FCN	:	total function calls
GLM1	:	third order GLM Set 1
GLM2	:	third order GLM Set 2
GLM3	:	third order GLM Set 3
RK(3)	:	third order calssical Runge-Kutta method from Butcher (2008)

- Problem 1.1

$$y' = -y$$
$$y(0) = 1, \qquad 0 \leq x \leq 10$$

Solution : $y(x) = e^{-x}$
Source : Zanariah and Suleiman (2011)

- Problem 1.2

$$y_1'' = -y_1(y_1^2 + y_2^2)^{-\frac{3}{2}},\ y_2'' = -y_2(y_1^2 + y_2^2)^{-\frac{3}{2}},$$
$$y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = 0,\ y_2'(0) = 1,\ 0 \leq x \leq 1.$$

The first order scheme is

$$y_1 = y_3,$$
$$y_2 = y_4,$$
$$y_3 = -y_1(y_1^2 + y_2^2)^{-\frac{3}{2}},$$
$$y_4 = -y_2(y_1^2 + y_2^2)^{-\frac{3}{2}},$$
$$y_1(0) = 1,\ y_2(0) = 0,\ y_3(0) = 0,\ y_4(0) = 1,\ 0 \leq x \leq 1.$$

Solution :
$y_1(x) = cos(x),\ y_2(x) = sin(x),\ y_3(x) = -sin(x),\ y_4(x) = cos(x).$
Source : Hull et al. (1972)

GLM for ODEs

Numerical results

h	Methods	P1	P2
0.1	GLM1	1.033467(-6)	2.331140(-5)
	GLM2	4.924486(-6)	4.489148(-5)
	GLM3	6.743934(-6)	4.909702(-5)
	RK(3)	1.660682(-5)	1.063867(-4)
0.05	GLM1	1.735809(-7)	2.321687(-6)
	GLM2	6.286938(-7)	5.211417(-6)
	GLM3	7.365140(-7)	5.102407(-6)
	RK(3)	1.994295(-6)	1.363147(-5)
0.01	GLM1	1.643349(-9)	1.378402(-8)
	GLM2	5.095354(-9)	3.827213(-8)
	GLM3	5.260627(-9)	3.542277(-8)
	RK(3)	1.545145(-8)	1.112752(-7)
0.005	GLM1	2.091770(-10)	1.639213(-9)
	GLM2	6.378138(-10)	4.722891(-9)
	GLM3	6.480860(-10)	4.542803(-9)
	RK(3)	1.923719(-9)	1.394461(-8)
0.001	GLM1	1.697234(-12)	1.256535(-11)
	GLM2	5.108070(-12)	3.738154(-11)
	GLM3	5.130000(-12)	3.709081(-11)
	RK(3)	1.534058(-11)	1.117826(-10)

Graph for Problem 1

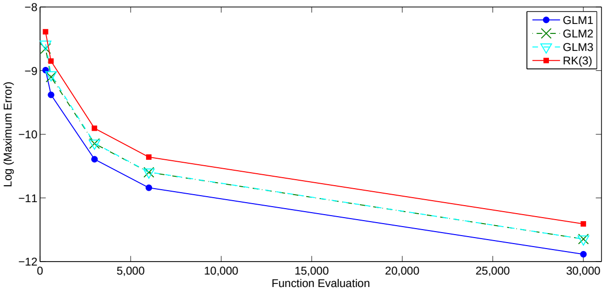


Figure 1. Graph of fuction evaluation versus maximum error for Problem 1

Graph for Problem 1

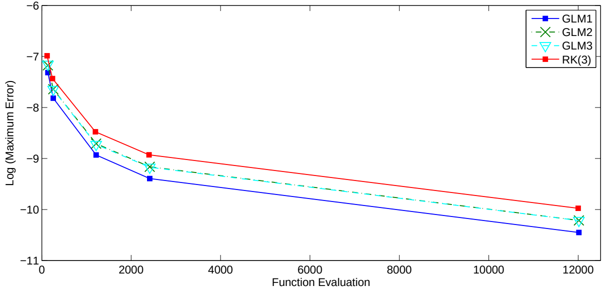


Figure 2. Graph of fuction evaluation versus maximum error for Problem 2

Discussion and Conclusion

- In this section, we derived the order conditions and gave the 3 new sets of coefficients for the third order General Linear Method.
- Then we applied the GLM on some test problems and compared the results with RK method.
- Numerical results showed that the GLM is more accurate than RK method in solving ODEs.

General Linear Method for Solving Volterra Integro-Differential Equtaions

SAWONA 2018, 3-4 April 2018

Volterra Integro-Differential Equations

- Volterra integro-differential equations (VIDEs) are widely used in our industry such as in electrodynamics, oscillation theory, and biomathematics problems.

Volterra Integro-Differential Equations

- Volterra integro-differential equations (VIDEs) are widely used in our industry such as in electrodynamics, oscillation theory, and biomathematics problems.
- Integro-differential equations contain differential and integral operators in the equation.

Volterra Integro-Differential Equations

- Volterra integro-differential equations (VIDEs) are widely used in our industry such as in electrodynamics, oscillation theory, and biomathematics problems.
- Integro-differential equations contain differential and integral operators in the equation.
- An integro-differential equation is an equation in which the unknown function $y(x)$ appears under the integral sign and contains an ordinary derivative $y^{(n)}(x)$.

Volterra Integro-Differential Equations

- Volterra integro-differential equations (VIDEs) are widely used in our industry such as in electrodynamics, oscillation theory, and biomathematics problems.
- Integro-differential equations contain differential and integral operators in the equation.
- An integro-differential equation is an equation in which the unknown function $y(x)$ appears under the integral sign and contains an ordinary derivative $y^{(n)}(x)$.
- A standard integro-differential equation is in the form of:

$$y^{(n)} = f(x) + \lambda \sum_{g(x)}^{h(x)} K(x,t)y(t)dt, \quad (1)$$

where $g(x)$ and $h(x)$ are the limit of the integration, λ is a constant parameter.$K(x,t)$ is the kernel of the integro equation The function of $f(x)$ and $K(x,t)$ are given in advance.

Volterra Integro-Differential Equations

- Consider the numerical solution of the second kind of volterra integro-differential (VIDEs) equation

$$y' = f(x) + \int_0^x k(x,t)y(t)dt, \; y(x_0) = y_0. \tag{2}$$

$y(x_n)$ denotes the exact value of y at $x_n = x_0 + h$. y_n to denote an approximation value of y at x_n.

Volterra Integro-Differential Equations

- Consider the numerical solution of the second kind of volterra integro-differential (VIDEs) equation

$$y' = f(x) + \int_0^x k(x,t)y(t)dt, \; y(x_0) = y_0. \tag{2}$$

$y(x_n)$ denotes the exact value of y at $x_n = x_0 + h$. y_n to denote an approximation value of y at x_n.

- In order to solve the integral part in (2), familiar numerical integration methods are used to approximate like Simpson Method I, Simpson Method II (Linz(1985)) and Lagrange interpolation polynomial (Filiz(2014)).

Definition 1 (Simpson's Method II)

If three-eighths rule is used at the upper end, on the points$t_(n-3), t_(n-2), t_(n-1), t_n$, we get the weights

n is even: as in Simpson I

n is odd:

$$
\begin{aligned}
&w_{n0} = \tfrac{1}{3}, && n \geq 5 \\
&w_{n,2i} = \tfrac{2}{3}, && i = 1, 2, \dots, \tfrac{n-5}{2}, \\
&w_{n,2i+1} = \tfrac{4}{3}, && i = 1, 2, \dots, \tfrac{n-5}{2}, \\
&w_{n,n-3} = \tfrac{17}{24} - \tfrac{1}{3}\delta_{n3}, \\
&w_{n,n-1} = w_{n,n-2} = \tfrac{9}{8}, \\
&w_{nn} = \tfrac{3}{8}.
\end{aligned}
$$

δ_{ij} denotes the Kronecker delta $\delta_{ij} = 0, i \neq j, \delta_{ii} = 0.$

Volterra Integro-Differential Equations

Definition 2 (Simpson's Method I)

If three-eighths rule is applied on the points t_0, t_1, t_2, t_3, one gets the weights (for $n \geq 2$)

n is even:
$$\begin{aligned} w_{n0} &= w_{nn} = \tfrac{1}{3}, \\ w_{n,2i} &= \tfrac{2}{3}, \qquad i = 0, 1, \dots, \tfrac{n}{2} - 1, \\ w_{n,2i+1} &= \tfrac{4}{3}, \qquad i = 0, 1, \dots, \tfrac{n}{2} - 1. \end{aligned}$$

n is odd:
$$\begin{aligned} w_{n0} &= \tfrac{3}{8}, \\ w_{n1} &= w_{n2} = \tfrac{9}{8}, \\ w_{n3} &= \tfrac{17}{24} - \tfrac{1}{3}\delta_{n3}, \\ w_{n,2i} &= \tfrac{4}{3}, \qquad i = 2, 3, \dots, \tfrac{n-1}{2}, \\ w_{n,2i+1} &= \tfrac{2}{3}, \qquad i = 2, 3, \dots, \tfrac{n-3}{2}, \\ w_{nn} &= \tfrac{1}{3}, \qquad n \geq 5. \end{aligned}$$

δ_{ij} denotes the Kronecker delta $\delta_{ij} = 0, i \neq j, \delta_{ii} = 0.$

Volterra Integro-Differential Equations

Definition 3 (Lagrange Interpolation Polynomial)

Polynomial $P(x)$ of degree $\neq (n-1)$ that pass through the n points$(x_1, y_1 = f(x_1)), (x_2, y_2 = f(x_2)), \ldots, (x_n, y_n = f(x_n))$, and is given by

$$P(x) = \sum_{j-1}^{n} P_j(x),$$

where

$$P_j(x) = y_j \prod_{k=1, k\neq 1}^{n} \frac{x - x_k}{x_j - x_k}.$$

Literature Review

In the previous work, a lot of numerical methods are presented by the researchers for solving the integro-differential equations.

Literature Review

In the previous work, a lot of numerical methods are presented by the researchers for solving the integro-differential equations.

- Raftari in (2010) solve the VIDE using Homotopy perturbation method along with trapezoidal rule for the integral operator and compared it with the finite difference method.

Literature Review

In the previous work, a lot of numerical methods are presented by the researchers for solving the integro-differential equations.

- Raftari in (2010) solve the VIDE using Homotopy perturbation method along with trapezoidal rule for the integral operator and compared it with the finite difference method.
- Zarebnia in (2010), the author considered the numerical solutions of VIDE by means of the Sinc collocation method where Sinc methods are direct solvers to integral equations.

Literature Review

In the previous work, a lot of numerical methods are presented by the researchers for solving the integro-differential equations.

- Raftari in (2010) solve the VIDE using Homotopy perturbation method along with trapezoidal rule for the integral operator and compared it with the finite difference method.
- Zarebnia in (2010), the author considered the numerical solutions of VIDE by means of the Sinc collocation method where Sinc methods are direct solvers to integral equations.
- Filiz in (2013) applied several Runge-Kutta methods with different order associated with trapezoidal rule and Simpson's rule to VIDE.

Literature Review

- Filiz (2014) used a higher order method which is the Runge-Kutta-Fehlberg method to solve the VIDE. He developed a new numerical routine for the integral part by using Lagrange interpolation and combination of various numerical quadrature rules to attain higher accuracy.

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

From (4), we let the integral part be

$$\int_0^x k(x,t)y(t)dt = z(t), \tag{3}$$

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

From (4), we let the integral part be

$$\int_0^x k(x,t)y(t)dt = z(t), \tag{3}$$

then the VIDE is given as

$$y'(x) = F(x, y(x), z(x)), y(0) = y_0. \tag{4}$$

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

From (4), we let the integral part be

$$\int_0^x k(x,t)y(t)dt = z(t), \tag{3}$$

then the VIDE is given as

$$y'(x) = F(x, y(x), z(x)), y(0) = y_0. \tag{4}$$

- The third order GLM is combined with appropriate numerical quadrature rule in order to approximate the integral part given in equation (7).

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

From (4), we let the integral part be

$$\int_0^x k(x,t)y(t)dt = z(t), \tag{3}$$

then the VIDE is given as

$$y'(x) = F(x, y(x), z(x)), y(0) = y_0. \tag{4}$$

- The third order GLM is combined with appropriate numerical quadrature rule in order to approximate the integral part given in equation (7).
- To evaluate the integral $z(x)$, we use the combination of composite Simpson's II rule for the interval $[x_0, x_n]$

- Lagrange interpolation for interval $[x_n, x_{(n+c_i)}]$ at points $x = -1, x = 0, x = c_i$.

- Lagrange interpolation for interval $[x_n, x_{(n+c_i)}]$ at points $x = -1, x = 0, x = c_i$.
- Lagrange Interpolation:

$$c_2 = \frac{1}{2}, \qquad P(x) = -\frac{1}{72}y_1 + \frac{7}{24}y_2 + \frac{2}{9}y_3,$$

- Lagrange interpolation for interval $[x_n, x_{(n+c_i)}]$ at points $x = -1, x = 0, x = c_i$.
- Lagrange Interpolation:

$$c_2 = \frac{1}{2}, \qquad P(x) = -\frac{1}{72}y_1 + \frac{7}{24}y_2 + \frac{2}{9}y_3,$$

- The third order General Linear Method applied to approximate equation (8) on a set of interval $[0, X]$ of equally spaced grid points $x_0 < x_{1\ldots} < x_N = X$ where $0 \leq n \leq N$ with step size $h = \frac{X - x_0}{V}$ may be written as:

- Lagrange interpolation for interval $[x_n, x_{(n+c_i)}]$ at points $x = -1, x = 0, x = c_i$.
- Lagrange Interpolation:

$$c_2 = \frac{1}{2}, \qquad P(x) = -\frac{1}{72}y_1 + \frac{7}{24}y_2 + \frac{2}{9}y_3,$$

- The third order General Linear Method applied to approximate equation (8) on a set of interval $[0, X]$ of equally spaced grid points $x_0 < x_{1...} < x_N = X$ where $0 \leq n \leq N$ with step size $h = \frac{X - x_0}{V}$ may be written as:
- For interval $[x_0, x_n]$,
  if $n = 1$, trapezoidal rule is used
  then

$$z_n = \frac{h}{2}(K(x_{(n+c_i)}, x_{(n-1)})y_1(x_{(n-1)}) + (x_{(n+c_i)}, x_n)y_1(x_n))$$

,

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

else if n is even, composite $\frac{1}{3}$ Simpson's rule is used then

$$
\begin{aligned}
z_n &= \tfrac{h}{3}\Big(K(x_{n+c_i}, x_0)y_1(x_0) + 2\sum_{m=1}^{\frac{n}{2}-1} K(x_{n+c_i}, x_{2m})y_1(x_{2m}) + \\
&\quad 4\sum_{m=1}^{\frac{n}{2}} K(x_{n+c_i}, x_{2m-1})y_1(x_{2m-1}) + K(x_{n+c_i}, x_n)y_1(x_n)\Big),
\end{aligned}
$$

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

else if n is even, composite $\frac{1}{3}$ Simpson's rule is used then

$$z_n = \frac{h}{3}(K(x_{n+c_i}, x_0)y_1(x_0) + 2\sum_{m=1}^{\frac{n}{2}-1} K(x_{n+c_i}, x_{2m})y_1(x_{2m}) + 4\sum_{m=1}^{\frac{n}{2}} K(x_{n+c_i}, x_{2m-1})y_1(x_{2m-1}) + K(x_{n+c_i}, x_n)y_1(x_n)),$$

else if n is odd, composite Simpson's II rule is used then

$$z_n = \frac{h}{3}(K(x_{n+c_i}, x_0)y_1(x_0) + 2\sum_{m=1}^{\frac{n-3}{2}-1} K(x_{n+c_i}, x_{2m})y_1(x_{2m}) + 4\sum_{m=1}^{\frac{n-3}{2}} K(x_{n+c_i}, x_{2m-1})y_1(x_{2m-1}) + K(x_{n+c_i}, x_{n-3})y_1(x_{n-3}) + \frac{3h}{8}(K(x_{n+c_i}, x_{n-3})y_1(x_{n-3})) + 3K(x_{n+c_i}, x_{n-2})y_1(x_{n-2}) +$$

Therefore, General Linear Method after employing composite Simpson's rule and Lagrange interpolation is given as follows

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

Therefore, General Linear Method after employing composite Simpson's rule and Lagrange interpolation is given as follows

$$Y_1 = u_{11}y_1(x_n) + u_{12}y_2(x_n),$$

$$Y_2 = a_{21}hF(x_n, Y_1, z_n) + u_{21}y_1(x_n) + u_{22}y_2(x_n),$$

$$\begin{aligned} z_{n+\frac{1}{2}} &= z_n + h(-\frac{1}{72}K(x_{n+\frac{1}{2}}, x_{n-1})y_1(x_{n-1}) - \frac{7}{24}K(x_{n+\frac{1}{2}}, x_n)y_1(x_n) \\ &\quad -\frac{2}{9}K(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}})y_1(x_{n+\frac{1}{2}})), \end{aligned}$$

$$Y_3 = a_{31}hF(x_n, Y_1, z_n) + a_{32}hF(x_{n+\frac{1}{2}}, Y_1, z_{n\frac{1}{2}}) + u_{31}y_1(x_n) + u_{32}y_2(x_n),$$

$$\begin{aligned} z_{n+1} &= z_n + h(-\frac{1}{12}K(x_{n+1}, x_{n-1})y_1(x_{n-1}) + \frac{2}{3}K(x_{n+1}, x_n)y_1(x_n) \\ &\quad +\frac{5}{12}K(x_{n+1}, x_{n+1})y_1(x_{n+1})), \end{aligned}$$

Implementation of General Linear Method for Solving Volterra Integro-Differential Equations

Therefore, General Linear Method after employing composite Simpson's rule and Lagrange interpolation is given as follows

$$Y_1 = u_{11}y_1(x_n) + u_{12}y_2(x_n),$$

$$Y_2 = a_{21}hF(x_n, Y_1, z_n) + u_{21}y_1(x_n) + u_{22}y_2(x_n),$$

$$\begin{aligned} z_{n+\frac{1}{2}} &= z_n + h(-\frac{1}{72}K(x_{n+\frac{1}{2}}, x_{n-1})y_1(x_{n-1}) - \frac{7}{24}K(x_{n+\frac{1}{2}}, x_n)y_1(x_n) \\ &\quad -\frac{2}{9}K(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}})y_1(x_{n+\frac{1}{2}})), \end{aligned}$$

$$Y_3 = a_{31}hF(x_n, Y_1, z_n) + a_{32}hF(x_{n+\frac{1}{2}}, Y_1, z_{n\frac{1}{2}}) + u_{31}y_1(x_n) + u_{32}y_2(x_n),$$

$$\begin{aligned} z_{n+1} &= z_n + h(-\frac{1}{12}K(x_{n+1}, x_{n-1})y_1(x_{n-1}) + \frac{2}{3}K(x_{n+1}, x_n)y_1(x_n) \\ &\quad +\frac{5}{12}K(x_{n+1}, x_{n+1})y_1(x_{n+1})), \end{aligned}$$

$$\begin{aligned} Y_1(x_{n+1}) &= b_{11}hF(x_n, Y_1, z_n) + b_{12}hF(x_{n+\frac{1}{2}}, Y_2, z_{n\frac{1}{2}}) + \\ &\quad b_{13}hF(x_{n+1}, y_3, z_{n+1}) + v_{11}y_1(x_n) + v_{12}y_2(x_n), \end{aligned}$$

Numerical Examples

In order to illustrate the efficiency and accuracy of the method, we solved some numerical tests. The exact solution $y(x)$ is used to estimate the global error. The following problems are solved by using the proposed method.

Below is the notation that we used in the table

h	:	Stepsize
MAXE	:	Maximum global error
RK3,	:	Third-order classical Runge-Kutta method with 3 stages.

Numerical Examples

Problem 1:

$$y(x)' = 1 + \int_0^x y(t)dt,\, t \geq 0,\, y(0) = y_0.$$

$$y(x) = sinh(x).$$

Numerical Examples

h	GLM3	RK3
	MAXE	
0.1	1.2347 $\times 10^{-6}$	4.7137$\times 10^{-6}$
0.025	9.9859$\times 10^{-9}$	7.1772$\times 10^{-8}$
0.01	5.6041$\times 10^{-10}$	4.6094$\times 10^{-9}$
0.005	6.7079$\times 10^{-11}$	5.7715$\times 10^{-10}$
0.001	5.1485$\times 10^{-13}$	4.6243$\times 10^{-12}$

Table 1: Maximum global error for Problem 1

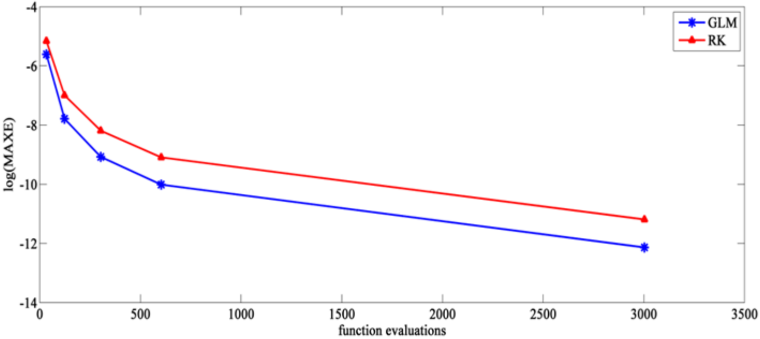


Figure 1: Log_{10}(Maximum Global Error) vs Function Evaluation for Problem 1

Numerical Examples

Problem 2:

$$y(x)' = 1 + 2x - y(x) - \int_0^x x(1+2x)e^{t(x-t)}y(t)dt,\ t \geq 0,\ y(0) = 1.$$

$$y(x) = e^{x^2}.$$

h	GLM3	RK3
	MAXE	
0.1	3.9332 $\times 10^{-6}$	4.8141$\times 10^{-5}$
0.025	1.4323$\times 10^{-7}$	1.4400$\times 10^{-6}$
0.01	1.0939$\times 10^{-8}$	4.6094$\times 10^{-7}$
0.005	1.4325$\times 10^{-9}$	5.7715$\times 10^{-8}$
0.001	1.1851$\times 10^{-11}$	4.6243$\times 10^{-10}$

Table 2: Maximum global error for Problem 2

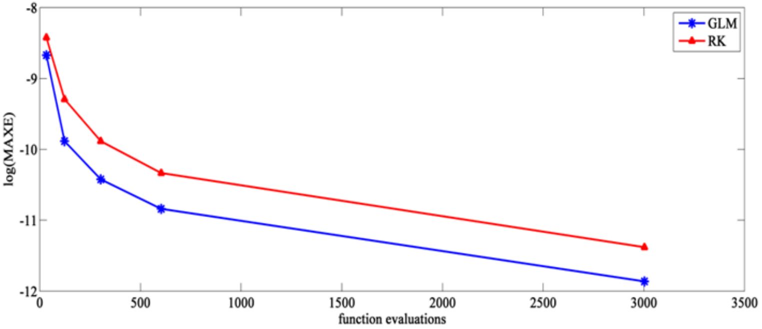


Figure 2: Log_{10}(Maximum Global Error) vs Function Evaluation for Problem 1

Problem 3:

$$y(x)' = ln(1+x)(\frac{x}{2}ln(1+x)+1) + \frac{1}{1+x} + y(x) - \int_0^x \frac{x}{t+1} y(t)dt,$$

$$y(0) = 0,$$

$$y(x) = ln(x+1).$$

Numerical Examples

h	GLM3	RK3
	MAXE	
0.1	6.3429 $\times 10^{-6}$	3.3432$\times 10^{-5}$
0.025	4.4142$\times 10^{-8}$	6.3237$\times 10^{-7}$
0.01	4.0164$\times 10^{-9}$	4.1259$\times 10^{-8}$
0.005	5.4245$\times 10^{-10}$	5.1811$\times 10^{-9}$
0.001	4.5689$\times 10^{-12}$	4.1572$\times 10^{-11}$

Table 3: Maximum global error for Problem 3

Numerical Examples

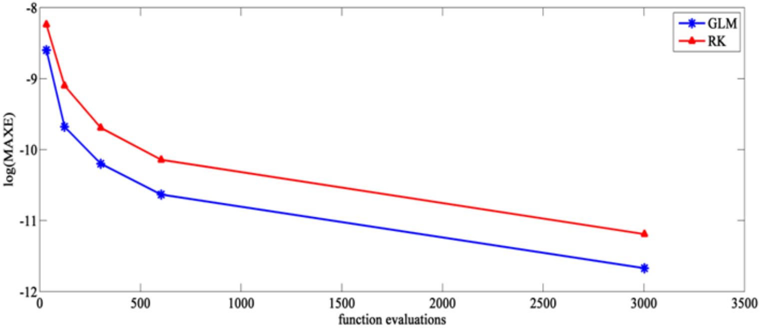


Figure 3: Log_{10}(Maximum Global Error) vs Function Evaluation for Problem 1

Discussion and Conclusion

- In this research, we derived the order conditions and gave a new set of coefficient for the third order General Linear Method.
- Handling the integral operator in Volterra integro-differential equations using Lagrange interpolation is demonstrated as well.
- Then we applied the GLM on some test problems and compared the results with RK method.
- Numerical results showed that the GLM is more accurate than RK method in solving VIDE.

General Linear Method for solving Fuzzy Differential Equations

SAWONA 2018, 3-4 April 2018

Fuzzy Differential Equations

- Fuzzy numbers generalize classical real numbers and a fuzzy number is a subset of the real line that has some additional properties.

Fuzzy Differential Equations

- Fuzzy numbers generalize classical real numbers and a fuzzy number is a subset of the real line that has some additional properties.
- **Definition 1** Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0, 1]$.

  Then u is a fuzzy number if it satisfies the following properties:

  1. u is normal, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
  2. u is a convex fuzzy set$(u(ts) + (1 - t)r) \geq min\{u(s), u(r)\}, \forall t \in [0, 1], x, r \in \mathbb{R}$,
  3. u is upper semicontinuous on $\mathbb{R}$
  4. $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Fuzzy Differential Equations

- Fuzzy numbers generalize classical real numbers and a fuzzy number is a subset of the real line that has some additional properties.
- **Definition 1** Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0, 1]$.

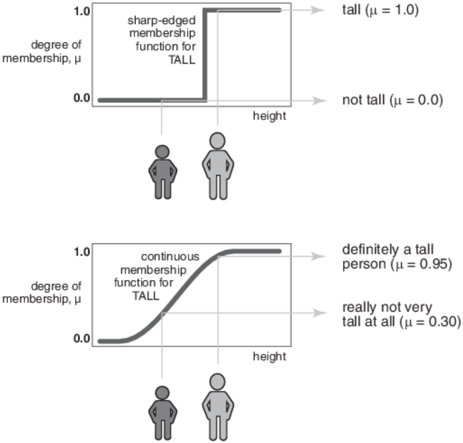
  Then u is a fuzzy number if it satisfies the following properties:

  1. u is normal, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
  2. u is a convex fuzzy set$(u(ts) + (1 - t)r) \geq min\{u(s), u(r)\}, \forall t \in [0, 1], x, r \in \mathbb{R}$,
  3. u is upper semicontinuous on $\mathbb{R}$
  4. $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

  $\mathbb{R}_F$ is called the space of fuzzy numbers. Then for $0 \leq r \leq 1$ if $u \in \mathbb{R}_F$, the r-level set $[u]_\alpha = \{x \in \mathbb{R} | u(x) \geq r\}$ is nonempty closed bounded interval. The r-level set of u is denoted by $[u]_r = [u_r^-, u_r^+]$.

- **Proposition 1** A fuzzy number u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \to \mathbb{R}$, defining the end-points of the r-cuts, satisfying the conditions:
  1. $u^- : r \to u_r^- \in \mathbb{R}$ is a bounded monotonic nondecreasing left-continuous function $\forall r \in (0,1]$ and right-continuous for $r = 0$;
  2. $u^+ : r \to u_r^+ \in \mathbb{R}$ is a bounded monotonic nonincreasing left-continuous function $\forall r \in (0,1]$ and right-continuous for $r = 0$;
  3. $u_1^- \leq u_1^+$ for $r = 1$, which implies $u_r^- \leq u_r^+ \; \forall r \in [0,1]$.

Difference between Crisp Set and Fuzzy Set



Fuzzy Differential Equations

The first order fuzzy initial value problem of fuzzy differential equation (FDE) defined by

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \in \mathbb{R}_F, \tag{1}$$

where function $f : \mathbb{R} \times \mathbb{R}_F \to \mathbb{R}_F$ is continuous and y_0 is a fuzzy number.

B. Bede, S. and G. Gal (2005)

They introduced a new generalized concept of differentiability which extends the previous Hukuhara derivative in order to overcome the solutions of becoming fuzzier.

Literature Review

B. Bede, S. and G. Gal (2005)

They introduced a new generalized concept of differentiability which extends the previous Hukuhara derivative in order to overcome the solutions of becoming fuzzier.

J. J. Nieto, A. Khastan, and K. Ivaz (2009)

Presented the generalized characterization theorem for the solutions of FDEs under generalized differentiability which permits the conversion of FDEs into a system of (ODEs)

Literature Review

B. Bede, S. and G. Gal (2005)

They introduced a new generalized concept of differentiability which extends the previous Hukuhara derivative in order to overcome the solutions of becoming fuzzier.

J. J. Nieto, A. Khastan, and K. Ivaz (2009)

Presented the generalized characterization theorem for the solutions of FDEs under generalized differentiability which permits the conversion of FDEs into a system of (ODEs)

B. Bede (2013)

A comprehensive study on fuzzy logic and fuzzy sets are found in here.

Definition 2 Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in [a, b]$. f is said to be strongly generalized Hukuhara differentiable at t_0 if there exists element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that

i for all $h > 0$ sufficiently small, $\exists\, f(t_0 + h) \ominus_H f(t_0), f(t_0) \ominus_H f(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(t_0 + h) \ominus_H f(t_0)}{h} = \lim_{h \searrow 0} \frac{f(t_0) \ominus_H f(t_0 - h)}{h} = f'_G(t_0), \tag{2}$$

is the (i)-differentiability on (a, b),

ii for all $h > 0$ sufficiently small, $\exists\, f(t_0) \ominus_H f(t_0 + h), f(t_0 - h) \ominus_H f(t_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(t_0) \ominus_H f(t_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(t_0 - h) \ominus_H f(t_0)}{-h} = f'_G(t_0). \tag{3}$$

is the (ii)-differentiability on (a, b),

Theorem 1:
Let $f : [a, b] \rightarrow \mathbb{R}_F$ and $x_0 \in [a, b]$ with $f_r^-(t)$ and $f_r^+(t)$ both differentiable at t. We say that

- if f is (i)-differentiable then
  $[f'(t)]_r = [(f_r^-)'(t), (f_r^+)'(t)], \qquad \forall r \in [0, 1]$
- if f is (ii)- differentiable then
  $[f'(t)]_r = [(f_r^+)'(t), (f_r^-)'(t)], \qquad \forall r \in [0, 1]$

Theorem 2 (Characterization theorem):
Consider the FIVP (1.2) where $f : (a, b) \times \mathbb{R}_F \to \mathbb{R}_F$ such that

i $[f(t, y)]_r = [f_r^-(t, y_r^-, y_r^+), f_r^+(t, y_r^-, y_r^+)]$,

ii f_r^- and f_r^+ are equicontinuous,

iii there exists $L > 0$ such that

$$|f_r^-(t, y^-, y^+) - f_r^-(t, z^-, z^+)| \leq L\max\{|y^- - z^-|, |y^+ - z^+|\},$$
$$|f_r^+(t, y^-, y^+) - f_r^+(t, z^-, z^+)| \leq L\max\{|y^- - z^-|, |y^+ - z^+|\},$$

where $\forall r \in [0, 1]$.

Hence from theorem 1, the FIVP (1) is equivalent to the following two system of ODEs:

$$\begin{cases} (y_r^-)'(t) = f_r^-(t, y_r^-, y_r^+) = F(t, y_r^-, y_r^+), & y_r^-(0) = (y_0)_r^-, \\ (y_r^+)'(t) = f_r^+(t, y_r^-, y_r^+) = G(t, y_r^-, y_r^+), & y_r^+(0) = (y_0)_r^+, \end{cases} \tag{4}$$

$$\begin{cases} (y_r^-)'(t) = f_r^+(t, y_r^-, y_r^+) = G(t, y_r^-, y_r^+), & y_r^-(0) = (y_0)_r^-, \\ (y_r^+)'(t) = f_r^-(t, y_r^-, y_r^+) = F(t, y_r^-, y_r^+), & y_r^+(0) = (y_0)_r^+, \end{cases} \tag{5}$$

Then, for type (i)-differentiability the FIVP (1) and system of ODEs (5) are equivalent and for type (ii)-differentiability the FIVP (1) and system of ODEs (6) are equivalent.

The fuzzy General Linear Method is represented by

$$y_i^-(t_{n+1};r) = \sum_{j=1}^{s=3} b_{ij} h F_j(t_n, y(t_n;r)) + \sum_{j=1}^{r=2} v_{ij} y_j^-(t_n;r), \quad i = 1, \ldots, r, \tag{6}$$

$$y_i^+(t_{n+1};r) = \sum_{j=1}^{s=3} b_{ij} h G_j(t_n, y(t_n;r)) + \sum_{j=1}^{r=2} v_{ij} y_j^+(t_n;r), \quad i = 1, \ldots, r, \tag{7}$$

where

$$Y_1^-(y(t_n;r)) = u_{11}y_1^-(t_n;r) + u_{12}y_2^-(t_n;r)$$

$$Y_1^-(y(t_n;r)) = u_{11}y_1^-(t_n;r) + u_{12}y_2^-(t_n;r) \quad (8)$$

$$Y_2^-(y(t_n;r)) = a_{21}hF_1(t_n, y(t_n;r)) + u_{21}y_1^-(t_n;r) + u_{22}y_2^-(t_n;r)$$

$$Y_2^+(y(t_n;r)) = a_{21}hG_1(t_n, y(t_n;r)) + u_{21}y_1^+(t_n;r) + u_{22}y_2^+(t_n;r) \quad (9)$$

$$Y_3^-(y(t_n;r)) = a_{31}hF_1(t_n, y(t_n;r)) + a_{32}hF_2(t_n, y(t_n;r)) + u_{31}y_1^-(t_n;r) + u_{32}y_2^-(t_n;r)$$

$$Y_3^+(y(t_n;r)) = a_{31}hG_1(t_n, y(t_n;r)) + a_{32}hG_2(t_n, y(t_n;r)) + u_{31}y_1^+(t_n;r) + u_{32}y_2^+(t_n;r) \quad (10)$$

and

$$F_1(t_n, y(t_n; r)) = min\{f(t_n + c_1 h, u) | u \in [Y_1^-(y(t_n; r)), Y_1^+(y(t_n; r))]\},$$
$$G_1(t_n, y(t_n; r)) = max\{f(t_n + c_1 h, u) | u \in [Y_1^-(y(t_n; r)), Y_1^+(y(t_n; r))]\},$$
$$F_2(t_n, y(t_n; r)) = min\{f(t_n + c_2 h, u) | u \in [Y_2^-(y(t_n; r)), Y_2^+(y(t_n; r))]\},$$
$$G_2(t_n, y(t_n; r)) = max\{f(t_n + c_2 h, u) | u \in [Y_2^-(y(t_n; r)), Y_2^+(y(t_n; r))]\},$$
$$F_3(t_n, y(t_n; r)) = min\{f(t_n + c_3 h, u) | u \in [Y_3^-(y(t_n; r)), Y_3^+(y(t_n; r))]\},$$
$$G_3(t_n, y(t_n; r)) = max\{f(t_n + c_3 h, u) | u \in [Y_3^-(y(t_n; r)), Y_3^+(y(t_n; r))]\},$$

(7) and (8) are the fuzzy GLM derived based on type (i)-differentiability.

For under type (ii)-differentiability:

$$y_i^{-}(t_{n+1};r) = \sum_{j=1}^{s=3} b_{ij} h G_j(t_n, y(t_n;r)) + \sum_{j=1}^{r=2} v_{ij} y_j^{-}(t_n;r), \quad i = 1, \ldots, r, \tag{11}$$

$$y_i^{+}(t_{n+1};r) = \sum_{j=1}^{s=3} b_{ij} h F_j(t_n, y(t_n;r)) + \sum_{j=1}^{r=2} v_{ij} y_j^{+}(t_n;r), \quad i = 1, \ldots, r, \tag{12}$$

where

$$
\begin{aligned}
Y_1^-(y(t_n;r)) &= u_{11}y_1^-(t_n;r) + u_{12}y_2^-(t_n;r) \\
Y_1^+(y(t_n;r)) &= u_{11}y_1^+(t_n;r) + u_{12}y_2^+(t_n;r) \qquad (13) \\
Y_2^-(y(t_n;r)) &= a_{21}hG_1(t_n, y(t_n;r)) + u_{21}y_1^-(t_n;r) + u_{22}y_2^-(t_n;r) \\
Y_2^+(y(t_n;r)) &= a_{21}hF_1(t_n, y(t_n;r)) + u_{21}y_1^+(t_n;r) + u_{22}y_2^+(t_n;r) \qquad (14) \\
Y_3^-(y(t_n;r)) &= a_{31}hG_1(t_n, y(t_n;r)) + a_{32}hG_2(t_n, y(t_n;r)) \\
&\quad + u_{31}y_1^-(t_n;r) + u_{32}y_2^-(t_n;r) \\
Y_3^+(y(t_n;r)) &= a_{31}hF_1(t_n, y(t_n;r)) + a_{32}hF_2(t_n, y(t_n;r)) \\
&\quad + u_{31}y_1^+(t_n;r) + u_{32}y_2^+(t_n;r). \qquad (15)
\end{aligned}
$$

r	:	r-level set of fuzzy numbers
y^{-}	:	left bounded of approximate solution
y^{+}	:	right bounded of approximate solution
$\tilde{Y}^{-}$	:	left bounded of exact solution
$\tilde{Y}^{+}$	:	right bounded of exact solution
$E^{-}(t;r)$	:	left bounded error of approxiate solution
$E^{+}(t;r)$	:	right bounded error of approxiate solution
GLM(3)	:	fuzzy third order GLM Set 1
RK(3)	:	fuzzy third order classical Runge-Kutta method from (Butcher, 2008)
ABM(3)	:	Third order Adams-Bashforth-Moulton predictor-corrector from (Allahvinraloo et al., 2007)
ANN	:	Artificial neural network method from (Effati and Pakdaman, 2010)

$$y'(t) = y(t)$$
$$y(0) = [0.75 + 0.25r, 1.125 - 0.125r], \qquad 0 \leq t \leq 1$$

The equivalent system of ODEs based on (i)-differentiability:

$$(y^-)'(t;r) = y^-(t;r), \quad y^-(0;r) = 0.75 + 0.25r$$
$$(y^+)'(t;r) = y^+(t;r), \quad y^+(0;r) = 1.125 - 0.125r$$

Solutions :

$$\tilde{Y}^-(t;r) = (0.75 + 0.25r)e^t$$
$$\tilde{Y}^+(t;r) = (1.125 - 0.125r)e^t$$

Source : Ma et al. (1999)

Numerical results

Numerical results for Problem 1

	$t=0.5$			$t=1$			
	GLM(3)	**RK(3)**	**ABM(3)**	**GLM(3)**	**RK(3)**	**ABM(3)**	**ANN**
r	$E^{-}(0.5;r)$	$E^{-}(0.5;r)$	$E^{-}(0.5;r)$	$E^{-}(1;r)$	$E^{-}(1;r)$	$E^{-}(1;r)$	$E^{-}(1;r)$
0.0	3.735899(-6)	2.378365(-5)	6.323706(-5)	1.231826(-5)	7.842448(-5)	2.249512(-4)	8.895270(-5)
0.1	3.860429(-6)	2.457644(-5)	6.534496(-5)	1.272887(-5)	8.103863(-5)	2.324496(-4)	2.903725(-5)
0.2	3.984959(-6)	2.536923(-5)	6.745287(-5)	1.313947(-5)	8.365278(-5)	2.399480(-4)	4.693243(-5)
0.3	4.109488(-6)	2.616202(-5)	6.956077(-5)	1.355008(-5)	8.626693(-5)	2.474464(-4)	2.484654(-5)
0.4	4.234018(-6)	2.695481(-5)	7.166867(-5)	1.396069(-5)	8.888108(-5)	2.549447(-4)	4.739291(-6)
0.5	4.358548(-6)	2.774760(-5)	7.377657(-5)	1.437130(-5)	9.149523(-5)	2.624431(-4)	5.478406(-5)
0.6	4.483078(-6)	2.854039(-5)	7.588448(-5)	1.478191(-5)	9.410938(-5)	2.699415(-4)	2.827934(-5)
0.7	4.607608(-6)	2.933317(-5)	7.799238(-5)	1.519252(-5)	9.672353(-5)	2.774398(-4)	4.693161(-5)
0.8	4.732138(-6)	3.012596(-5)	8.010028(-5)	1.560313(-5)	9.933768(-5)	2.849382(-4)	9.712388(-5)
0.9	4.856668(-6)	3.091875(-5)	8.220818(-5)	1.601373(-5)	1.019518(-4)	2.924366(-4)	1.417971(-5)
1.0	4.981198(-6)	3.171154(-5)	8.431608(-5)	1.642434(-5)	1.045660(-4)	2.999350(-4)	5.574679(-5)
r	$E^{+}(0.5;r)$	$E^{+}(0.5;r)$	$E^{+}(0.5;r)$	$E^{+}(1;r)$	$E^{+}(1;r)$	$E^{+}(1;r)$	$E^{+}(1;r)$
0.0	5.603848(-6)	3.567548(-5)	9.485559(-5)	1.847739(-5)	1.176367(-4)	3.374268(-4)	6.003329(-5)
0.1	5.541583(-6)	3.527909(-5)	9.380164(-5)	1.827208(-5)	1.163296(-4)	3.336777(-4)	1.107843(-6)
0.2	5.479318(-6)	3.488269(-5)	9.274769(-5)	1.806678(-5)	1.150226(-4)	3.299285(-4)	2.582699(-5)
0.3	5.417053(-6)	3.448630(-5)	9.169374(-5)	1.786147(-5)	1.137155(-4)	3.261793(-4)	8.472236(-6)
0.4	5.354788(-6)	3.408991(-5)	9.063979(-5)	1.765617(-5)	1.124084(-4)	3.224301(-4)	3.384699(-5)
0.5	5.292523(-6)	3.369351(-5)	8.958584(-5)	1.745086(-5)	1.111013(-4)	3.186809(-4)	1.443122(-5)
0.6	5.230258(-6)	3.329712(-5)	8.853189(-5)	1.724556(-5)	1.097943(-4)	3.149317(-4)	4.194040(-5)
0.7	5.167993(-6)	3.290072(-5)	8.747794(-5)	1.704026(-5)	1.084872(-4)	3.111825(-4)	5.199824(-5)
0.8	5.105728(-6)	3.250433(-5)	8.642399(-5)	1.683495(-5)	1.071801(-4)	3.074333(-4)	3.025500(-5)
0.9	5.043463(-6)	3.210793(-5)	8.537004(-5)	1.662965(-5)	1.058730(-4)	3.036842(-4)	1.541195(-4)
1.0	4.981198(-6)	3.171154(-5)	8.431608(-5)	1.642434(-5)	1.045660(-4)	2.999350(-4)	2.787937(-5)

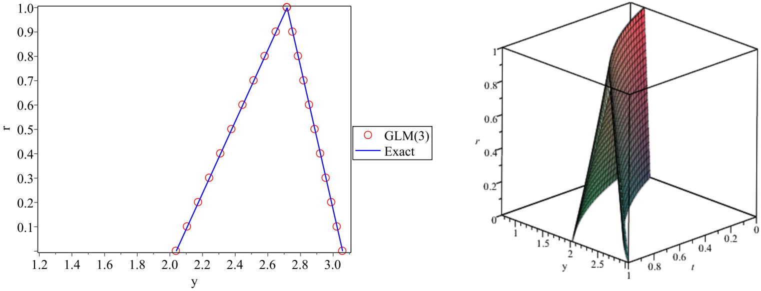


Figure 1(a)

Figure 1(b)

Figure 1. (a) circle-point: GLM(3), Line: Exact; (b) 3D-plot at $t = 1.0$

$y'(t) = -y(t),$
$y(0) = [r-1, 1-r], \qquad 0 \leq t \leq 1$

The equivalent system of ODEs based on (i)-differentiability:

$$(y^-)'(t;r) = -y^+(t;r), \quad y^-(0;r) = r-1$$
$$(y^+)'(t;r) = -y^-(t;r), \quad y^+(0;r) = 1-r$$

Solutions based on (i)-differentiability:

$$\tilde{Y}^-(t;r) = (r-1)e^t,$$
$$\tilde{Y}^+(t;r) = (1-r)e^t$$

The equivalent system of ODEs based on (ii)-differentiability:

$$(y^-)'(t;r) = -y^-(t;r), \quad y^-(0;r) = r - 1$$
$$(y^+)'(t;r) = -y^+(t;r), \quad y^+(0;r) = 1 - r$$

Solutions based on (ii)-differentiability:

$$\tilde{Y}^-(t;r) = (r-1)e^{-t},$$
$$\tilde{Y}^+(t;r) = (1-r)e^{-t}$$

Source : Nieto et al. (2009)

Numerical results

Numerical results for Problem 2 based on (i)-differentiability

	$t=0.5$			$t=1$		
	GLM(3)	**RK(3)**	**ABM(3)**	**GLM(3)**	**RK(3)**	**ABM(3)**
r	$E^-(0.5;r)$	$E^-(0.5;r)$	$E^-(0.5;r)$	$E^-(1;r)$	$E^-(1;r)$	$E^-(1;r)$
0.0	4.981198(-6)	3.171154(-5)	8.431608(-5)	1.642434(-5)	1.045660(-5)	2.999350(-4)
0.1	4.483078(-6)	2.854039(-5)	7.588448(-5)	1.478191(-5)	9.410938(-5)	2.699415(-4)
0.2	3.984959(-6)	2.536923(-5)	6.745287(-5)	1.313947(-5)	8.365278(-5)	2.399480(-4)
0.3	3.486839(-6)	2.219808(-5)	5.902126(-5)	1.149704(-5)	7.319618(-5)	2.099545(-4)
0.4	2.988719(-6)	1.902692(-5)	5.058965(-5)	9.854606(-6)	6.273959(-5)	1.799610(-4)
0.5	2.490599(-6)	1.585577(-5)	4.215804(-5)	8.212172(-6)	5.228299(-5)	1.499675(-4)
0.6	1.992479(-6)	1.268462(-5)	3.372643(-5)	6.569738(-6)	4.182639(-5)	1.199740(-4)
0.7	1.494359(-6)	9.513462(-6)	2.529482(-5)	4.927303(-6)	3.136979(-5)	8.998049(-5)
0.8	9.962396(-7)	6.342308(-6)	1.686322(-5)	3.284869(-6)	2.091319(-5)	5.998699(-5)
0.9	4.981198(-7)	3.171154(-6)	8.431608(-6)	1.642434(-6)	1.045660(-5)	2.999350(-5)
1.0	0	0	0	0	0	0
r	$E^+(0.5;r)$	$E^+(0.5;r)$	$E^+(0.5;r)$	$E^+(1;r)$	$E^+(1;r)$	$E^+(1;r)$
0.0	4.981198(-6)	3.171154(-5)	8.431608(-5)	1.642434(-5)	1.045660(-5)	2.999350(-4)
0.1	4.483078(-6)	2.854039(-5)	7.588448(-5)	1.478191(-5)	9.410938(-5)	2.699415(-4)
0.2	3.984959(-6)	2.536923(-5)	6.745287(-5)	1.313947(-5)	8.365278(-5)	2.399480(-4)
0.3	3.486839(-6)	2.219808(-5)	5.902126(-5)	1.149704(-5)	7.319618(-5)	2.099545(-4)
0.4	2.988719(-6)	1.902692(-5)	5.058965(-5)	9.854606(-6)	6.273959(-5)	1.799610(-4)
0.5	2.490599(-6)	1.585577(-5)	4.215804(-5)	8.212172(-6)	5.228299(-5)	1.499675(-4)
0.6	1.992479(-6)	1.268462(-5)	3.372643(-5)	6.569738(-6)	4.182639(-5)	1.199740(-4)
0.7	1.494359(-6)	9.513462(-6)	2.529482(-5)	4.927303(-6)	3.136979(-5)	8.998049(-5)
0.8	9.962396(-7)	6.342308(-6)	1.686322(-5)	3.284869(-6)	2.091319(-5)	5.998699(-5)
0.9	4.981198(-7)	3.171154(-6)	8.431608(-6)	1.642434(-6)	1.045660(-5)	2.999350(-5)
1.0	0	0	0	0	0	0

Numerical results

Numerical results for Problem 2 based on (ii)-differentiability

	$t = 0.5$			$t = 1$		
	GLM(3)	**RK(3)**	**ABM(3)**	**GLM(3)**	**RK(3)**	**ABM(3)**
r	$E^-(0.5; r)$	$E^-(0.5; r)$	$E^-(0.5; r)$	$E^-(1; r)$	$E^-(1; r)$	$E^-(1; r)$
0.0	8.520483(-7)	1.369017(-5)	3.852918(-5)	1.033467(-6)	1.660682(-5)	5.046724(-5)
0.1	7.668435(-7)	1.232115(-5)	3.467626(-5)	9.301205(-7)	1.494614(-5)	4.542052(-5)
0.2	6.816386(-7)	1.095213(-5)	3.082334(-5)	8.267738(-7)	1.328546(-5)	4.037379(-5)
0.3	5.964338(-7)	9.583117(-6)	2.697042(-5)	7.234270(-7)	1.162478(-5)	3.532707(-5)
0.4	5.112290(-7)	8.214100(-6)	2.311751(-5)	6.200803(-7)	9.964094(-6)	3.028035(-5)
0.5	4.260241(-7)	6.845083(-6)	1.926459(-5)	5.167336(-7)	8.303412(-6)	2.523362(-5)
0.6	3.408193(-7)	5.476067(-6)	1.541167(-5)	4.133869(-7)	6.642730(-6)	2.018690(-5)
0.7	2.556145(-7)	4.107050(-6)	1.155875(-5)	3.100402(-7)	4.982047(-6)	1.514017(-5)
0.8	1.704097(-7)	2.738033(-6)	7.705835(-6)	2.066934(-7)	3.321365(-6)	1.009345(-5)
0.9	8.520483(-8)	1.369017(-6)	3.852918(-6)	1.033467(-7)	1.660682(-6)	5.046724(-6)
1.0	0	0	0	0	0	0
r	$E^+(0.5; r)$	$E^+(0.5; r)$	$E^+(0.5; r)$	$E^+(1; r)$	$E^+(1; r)$	$E^+(1; r)$
0.0	8.520483(-7)	1.369017(-5)	3.852918(-5)	1.033467(-6)	1.660682(-5)	5.046724(-5)
0.1	7.668435(-7)	1.232115(-5)	3.467626(-5)	9.301205(-7)	1.494614(-5)	4.542052(-5)
0.2	6.816386(-7)	1.095213(-5)	3.082334(-5)	8.267738(-7)	1.328546(-5)	4.037379(-5)
0.3	5.964338(-7)	9.583117(-6)	2.697042(-5)	7.234270(-7)	1.162478(-5)	3.532707(-5)
0.4	5.112290(-7)	8.214100(-6)	2.311751(-5)	6.200803(-7)	9.964094(-6)	3.028035(-5)
0.5	4.260241(-7)	6.845083(-6)	1.926459(-5)	5.167336(-7)	8.303412(-6)	2.523362(-5)
0.6	3.408193(-7)	5.476067(-6)	1.541167(-5)	4.133869(-7)	6.642730(-6)	2.018690(-5)
0.7	2.556145(-7)	4.107050(-6)	1.155875(-5)	3.100402(-7)	4.982047(-6)	1.514017(-5)
0.8	1.704097(-7)	2.738033(-6)	7.705835(-6)	2.066934(-7)	3.321365(-6)	1.009345(-5)
0.9	8.520483(-8)	1.369017(-6)	3.852918(-6)	1.033467(-7)	1.660682(-6)	5.046724(-6)
1.0	0	0	0	0	0	0

Numerical results

Graphs for Problem 2

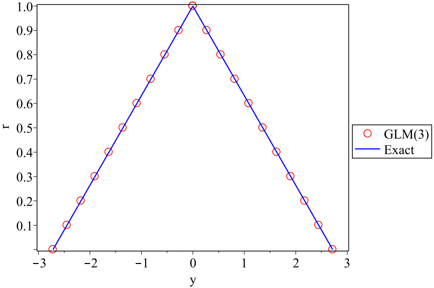


Figure 2(a)

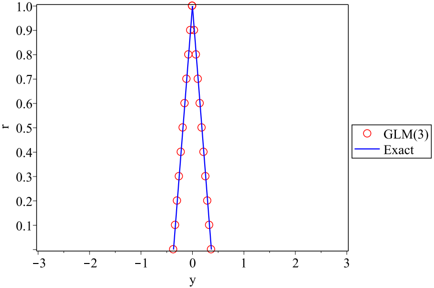


Figure 2(b)

Figure 2. (a) circle-point: GLM(3), Line: Exact with (i)-differentiability; (b) circle-point: GLM(3), Line: Exact with (ii)-differentiability

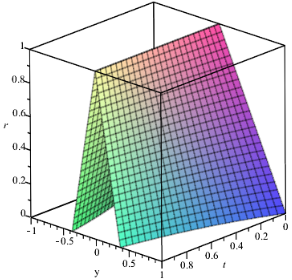


Figure 2(c)

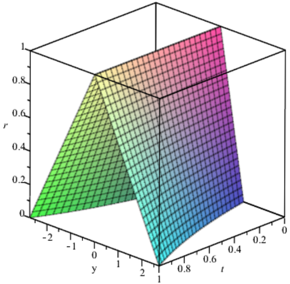


Figure 2(d)

Figure 2. (c) 3D-plot with (i)-differentiability; (b) 3D-plot with (ii)-differentiability

Conclusion

- The generalized Hukuhara differetiability was implemented to develop the fuzzy version of a third order GLM and tested on FDEs.

Conclusion

- The generalized Hukuhara differetiability was implemented to develop the fuzzy version of a third order GLM and tested on FDEs.
- From the numerical results, GLM is more accurate than the RK and Adam Bashforth method of same order.

General Linear Method for Solving Fuzzy Volterra Integro-Differential Equtaions

SAWONA 2018, 3-4 April 2018

Fuzzy Volterra Integro-Differential Equations

The first order fuzzy initial value problem of fuzzy volterra integro-differential equation (FVIDE) of the form

$$y'(t) = f(t, y) + \int_0^x K(t, s)y(s)ds, \quad y(t_0) = y_0 \in \mathbb{R}_F, \tag{1}$$

where function $f : \mathbb{R} \times \mathbb{R}_F \to \mathbb{R}_F$, crisp function $K(t, s)$ are continuous and y_0 is a fuzzy number.

Literature Review

P. Linz (1985)

Presented the study of Volterra integro-differential equations.

Literature Review

P. Linz (1985)

Presented the study of Volterra integro-differential equations.

A. Filiz (2013)

Presented the methods in finding the numerical results of Volterra integro-differential equations.

Literature Review

P. Linz (1985)

Presented the study of Volterra integro-differential equations.

A. Filiz (2013)

Presented the methods in finding the numerical results of Volterra integro-differential equations.

M. Matinfar, M. Ghanbari, and M. Nuraei (2013)

Demonstrated the approximation of solutions or fuzzy Volterra-integro differential equations using variational iteration method.

From the characterization theorem, the parametric forms of FVIDEs based on (i)-differentiability are as follow:

$$
\begin{aligned}
(y_r^-)'(t) &= f_r^-(t, y_r^-(t), y_r^+(t)) + \int_0^t K^-(t,s)y(s)ds \\
&= f_r^-\left(t, y_r^-(t), y_r^+(t), \int_0^t K^-(t,s)y(s)ds\right), \qquad (2) \\
(y_r^+)'(t) &= f_r^+(t, y_r^-(t), y_r^+(t)) + \int_0^t K^+(t,s)y(s)ds \\
&= f_r^+\left(t, y_r^-(t), y_r^+(t), \int_0^t K^+(t,s)y(s)ds\right), \quad 0 \leq s \leq t \leq 1 \qquad (3)
\end{aligned}
$$

where

$$
K^-(t,s)y(s) = \begin{cases} K(t,s)y_r^-(s), K(t,s) \geqslant 0, \\ K(t,s)y_r^+(s), K(t,s) \leq 0. \end{cases}
$$

$$
K^+(s,t)y(s) = \begin{cases} K(s,t)y_r^+(t), K(t,s) \geqslant 0, \\ K(s,t)y_r^-(t), K(t,s) \leq 0. \end{cases}
$$

In the integral part of the equation: $\int_0^t K(s,t)y(s)ds$,
the integration will be computed using Composite Simpson's Rule.
For even sub-intervals:

$$\int_0^{t_n} K(t,s)y(s)ds = K(t,t_0)y(t_0) + 2\sum_{j=1}^{\frac{n}{2}-1} K(t,t_{2m})y(t_{2m}) + 4\sum_{j=1}^{\frac{n}{2}} K(t,t_{2m-1})y(t_{2m-1}) + K(t,t_{2n})y(t_n) \quad (4)$$

For odd sub-intervals:

$$\int_0^{t_n} K(t,s)y(s)ds = (19) + \frac{3}{8} Simpson' srule \quad (5)$$

In the integral part of the equation: $\int_{t_n}^{t_{n+ch}} K(s,t)x(s)ds$,
on interval $[t_n, t_{n+\frac{1}{2}h}]$,

$$\int_{t_n}^{t_{n+\frac{1}{2}h}} K(t,s)y(s)ds = h\Big\{ -\frac{1}{72}K(t,s)y(t_{n-1}) + \frac{7}{24}K(t,s)y(t_n) + \frac{2}{9}K(t,s)y(t_{n+\frac{1}{2}h})\Big\}, \quad (6)$$

and on the interval $[t_n, t_{n+h}]$ we get

$$\int_{t_n}^{t_{n+h}} K(t,s)y(s)ds = h\Big\{ -\frac{1}{12}K(s,t)y(t_{n-1}) + \frac{2}{3}K(t,s)y(t_n) + \frac{5}{12}K(t,s)y(t_{n+h})\Big\}. \quad (7)$$

General Linear Method for Solving FVIDEs

Formulation

The fuzzy GLM for solving FVIDEs are similar as equations (7-11), however with extra integration operation as shown below:

$$F_1(t_n, y(t_n; r)) = min\,\{f(t_n + c_1 h, u, v) | u \in [Y_1^-(y(t_n; r)), Y_1^+(y(t_n; r))], v \in [z_1^-(y(t_n; r)), z_1^+(y(t_n; r))]\},$$

$$G_1(t_n, y(t_n; r)) = max\,\{f(t_n + c_1 h, u, v) | u \in [Y_1^-(y(t_n; r)), Y_1^+(y(t_n; r))]\}, v \in [z_1^-(y(t_n; r)), z_1^+(y(t_n; r))]$$

$$F_2(t_n, y(t_n; r)) = min\,\{f(t_n + c_2 h, u, v) | u \in [Y_2^-(y(t_n; r)), Y_2^+(y(t_n; r))]\}, v \in [z_2^-(y(t_n; r)), z_2^+(y(t_n; r))]$$

$$G_2(t_n, y(t_n; r)) = max\,\{f(t_n + c_2 h, u, v) | u \in [Y_2^-(y(t_n; r)), Y_2^+(y(t_n; r))]\}, v \in [z_2^-(y(t_n; r)), z_2^+(y(t_n; r))]$$

$$F_3(t_n, y(t_n; r)) = min\,\{f(t_n + c_3 h, u, v) | u \in [Y_3^-(y(t_n; r)), Y_3^+(y(t_n; r))]\}, v \in [z_3^-(y(t_n; r)), z_3^+(y(t_n; r))]$$

$$G_3(t_n, y(t_n; r)) = max\,\{f(t_n + c_3 h, u, v) | u \in [Y_3^-(y(t_n; r)), Y_3^+(y(t_n; r))]\}, v \in [z_3^-(y(t_n; r)), z_3^+(y(t_n; r))],$$

where

$$z_1^-(y(t_n)) = \int_0^{t_n} K^-(t,s)y(s;r)ds$$

$$z_1^+(y(t_n)) = \int_0^{t_n} K^+(t,s)y(s;r)ds$$

$$z_2^-(y(t_n)) = \int_0^{t_n} K^-(t,s)y(s;r)ds + \int_{t_n}^{t_{n+c_2 h}} K^-(t,s)y(s)ds$$

$$z_2^+(y(t_n)) = \int_0^{t_n} K^+(t,s)y(s;r)ds + \int_{t_n}^{t_{n+c_2 h}} K^+(t,s)y(s)ds$$

$$z_3^-(y(t_n)) = \int_0^{t_n} K^-(t,s)y(s)ds + \int_{t_n}^{t_{n+c_3 h}} K^-(t,s)y(s)ds$$

$$z_3^+(y(t_n)) = \int_0^{t_n} K^+(t,s)y(s)ds + \int_{t_n}^{t_{n+c_3 h}} K^+(t,s)y(s)ds$$

r	:	r-level set of fuzzy numbers
y^-	:	left bounded of approximate solution
y^+	:	right bounded of approximate solution
$\tilde{Y}^-$	:	left bounded of exact solution
$\tilde{Y}^+$	:	right bounded of exact solution
$E^-(t;r)$	:	left bounded error of approxiate solution
$E^+(t;r)$	:	right bounded error of approxiate solution
GLM(3)	:	fuzzy third order GLM Set 1
RK(3)	:	fuzzy third order classical Runge-Kutta method from (Butcher, 2008)

$$y'(t) = C\frac{1}{12t}(36 - 5t^4) + \int_0^t (t^2 + s^2)y(s;r)ds$$

$$C = [(r^5 + 2r)t^3, (6 - 3r^3)t^3], \quad y(0) = [0, 0], \qquad 0 \le s \le t \le 1$$

The equivalent system of ODEs based on (i)-differentiability:

$$(y^-)'(t;r) = \frac{1}{12}rt^2(r^4 + 2)(36 - 5t^4) + \int_0^t (t^2 + s^2)y^-(s;r), \quad y^-(0;r) = 0$$

$$(y^+)'(t;r) = \frac{1}{4}t^2(r^3 - 2)(5t^4 - 36) + \int_0^t (t^2 + s^2)y^+(s;r), \quad y^+(0;r) = 0$$

Solutions :

$$\tilde{Y}^-(t;r) = (r^5 + 2r)t^3$$

$$\tilde{Y}^+(t;r) = (6 - 3r^3)t^3$$

Source : Matinfar et al. (2013)

Numerical results

Problem 1

	$t = 0.5$		$t = 1$	
	GLM(3)	**RK(3)**	**GLM(3)**	**RK(3)**
r	$E^-(0.5; r)$	$E^-(0.5; r)$	$E^-(1; r)$	$E^-(1; r)$
0.0	0	0	0	0
0.1	9.956810(-11)	4.823713(-10)	1.209550(-9)	7.462263(-9)
0.2	1.992855(-10)	9.654662(-10)	2.420914(-9)	1.493572(-8)
0.3	2.998988(-10)	1.452902(-9)	3.643163(-9)	2.247632(-8)
0.4	4.033500(-10)	1.954086(-9)	4.899886(-9)	3.022960(-8)
0.5	5.133730(-10)	2.487103(-9)	6.236430(-9)	3.847537(-8)
0.6	6.360880(-10)	3.081621(-9)	7.727180(-9)	4.767250(-8)
0.7	7.806100(-10)	3.781770(-9)	9.482820(-8)	5.850381(-8)
0.8	9.596280(-10)	4.649055(-9)	1.165754(-8)	7.192064(-8)
0.9	1.190025(-9)	5.765231(-9)	1.445635(-8)	8.918787(-8)
1.0	1.493445(-9)	7.235207(-9)	1.814234(-8)	1.119283(-7)
r	$E^+(0.5; r)$	$E^+(0.5; r)$	$E^+(1; r)$	$E^+(1; r)$
0.0	2.986892(-9)	1.447042(-8)	3.628465(-8)	2.238567(-7)
0.1	2.985401(-9)	1.446318(-8)	3.626651(-8)	2.237446(-7)
0.2	2.974945(-9)	1.441254(-8)	3.613950(-8)	2.229612(-7)
0.3	2.946571(-9)	1.427506(-8)	3.579484(-8)	2.208346(-7)
0.4	2.891310(-9)	1.400736(-8)	3.512354(-8)	2.166932(-7)
0.5	2.800214(-9)	1.356601(-8)	3.401690(-8)	2.098656(-7)
0.6	2.664307(-9)	1.290761(-8)	3.236592(-8)	1.996801(-7)
0.7	2.474641(-9)	1.198874(-8)	3.006184(-8)	1.854652(-7)
0.8	2.222250(-9)	1.076599(-8)	2.699583(-8)	1.665493(-7)
0.9	1.898170(-9)	9.195954(-8)	2.305891(-8)	1.422609(-7)
1.0	1.493445(-9)	7.235207(-8)	1.814234(-8)	1.119283(-7)

Numerical results

Graphs for Problem 1

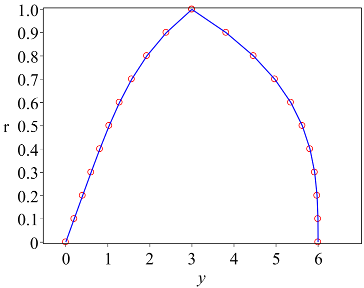


Figure 1(a)

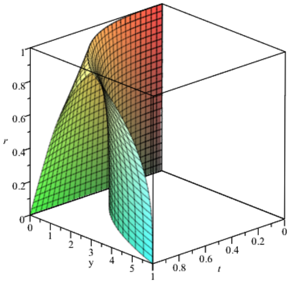


Figure 1(b)

Figure 1. (a) Circle-point: GLM(3), line: Exact; (b) 3D-plot at $t = 1.0$

Numerical results

Problem 2

$y'(t) = C + \int_0^t (-y(s))ds, \quad C = [2(r-2)sin(t), 2(2-3r)sin(t)],$
$y(0) = [3r-2, 2-r], \qquad 0 \le t \le 1$
The equivalent system of ODEs based on (i)-differentiability:

$$(y^-)'(t;r) = 2(r-2)sin(t) + \int_0^1 (-1)y^+(s;r), \quad y^-(0;r) = 3r-2$$
$$(y^+)'(t;r) = 2(2-3r)sin(t) + \int_0^1 (-1)y^-(s;r), \quad y^+(0;r) = 2-r$$

Solutions based on (i)-differentiability:

$$\tilde{Y}^-(t;r) = -rtsin(t) + (2-r)cos(t) + 2(r-1)(exp(t) + exp(-t))$$
$$\tilde{Y}^+(t;r) = -rtsin(t) + (3r-2)cos(t) + 2(1-r)(exp(t) + exp(-t))$$

Numerical results
Problem 2

The equivalent system of ODEs based on (ii)-differentiability:

$$(y^{-})'(t;r) = 2(2-3r)sin(t) + \int_0^1 (-1)y^{-}(s;r), \quad y^{-}(0;r) = 3r-2$$

$$(y^{+})'(t;r) = 2(r-2)sin(t) + \int_0^1 (-1)y^{+}(s;r), \quad y^{+}(0;r) = 2-r$$

Solutions based on (ii)-differentiability:

$$\tilde{Y}^{-}(t;r) = (3r-2)(cos(t) - tsin(t)), \quad \tilde{Y}^{+}(t;r) = (2-r)(cos(t) - tsin(t))$$

Source : Ghanbari (2016)

Numerical results

Numerical results for Problem 2 based on (i)-differentiability

	$t = 0.5$		$t = 1$	
	GLM(3)	**RK(3)**	**GLM(3)**	**RK(3)**
r	$E^-(0.5; r)$	$E^-(0.5; r)$	$E^-(1; r)$	$E^-(1; r)$
0.0	9.591260(-9)	3.884529(-8)	1.703350(-8)	8.745231(-8)
0.1	8.252610(-9)	3.333684(-8)	1.499912(-8)	7.665465(-8)
0.2	6.913930(-9)	2.782840(-8)	1.296471(-8)	6.585705(-8)
0.3	5.575350(-9)	2.231994(-8)	1.093031(-8)	5.505935(-8)
0.4	4.236740(-9)	1.681141(-8)	8.895910(-9)	4.426157(-8)
0.5	2.898090(-9)	1.130299(-8)	6.861520(-9)	3.346385(-8)
0.6	1.559467(-9)	5.794547(-9)	4.827130(-9)	2.266616(-8)
0.7	2.208310(-10)	2.860890(-10)	2.792740(-9)	1.186848(-8)
0.8	1.117786(-9)	5.222359(-9)	7.583400(-9)	1.070790(-8)
0.9	2.456414(-9)	1.073082(-8)	1.276027(-9)	9.726932(-8)
1.0	3.795042(-9)	1.623928(-8)	3.310421(-9)	2.052464(-8)
r	$E^+(0.5; r)$	$E^+(0.5; r)$	$E^+(1; r)$	$E^+(1; r)$
0.0	9.591260(-9)	3.884529(-8)	1.703350(-8)	8.745231(-8)
0.1	9.011600(-9)	3.658467(-8)	1.566117(-8)	8.075957(-8)
0.2	8.431990(-9)	3.432415(-8)	1.428888(-8)	7.406693(-8)
0.3	7.852400(-9)	3.206350(-8)	1.291656(-8)	6.737409(-8)
0.4	7.272740(-9)	2.980293(-8)	1.154423(-8)	6.068132(-8)
0.5	6.693130(-9)	2.754225(-8)	1.017195(-8)	5.398850(-8)
0.6	6.113500(-9)	2.528165(-8)	8.799650(-9)	4.729569(-8)
0.7	5.533890(-9)	2.302113(-8)	7.427330(-9)	4.060305(-8)
0.8	4.954250(-9)	2.076049(-8)	6.054989(-9)	3.391018(-8)
0.9	4.374664(-9)	1.849986(-8)	4.682732(-9)	2.721740(-8)
1.0	3.795042(-9)	1.623928(-8)	3.310421(-9)	2.052464(-8)

Numerical results

Numerical results for Problem 2 based on (ii)-differentiability

	$t = 0.5$		$t = 1$	
	GLM(3)	**RK(3)**	**GLM(3)**	**RK(3)**
r	$E^-(0.5;r)$	$E^-(0.5;r)$	$E^-(1;r)$	$E^-(1;r)$
0.0	7.590100(-9)	3.247856(-8)	6.620856(-9)	4.104928(-8)
0.1	6.451570(-9)	2.760677(-8)	5.627722(-9)	3.489188(-8)
0.2	5.313055(-9)	2.273503(-8)	4.634590(-9)	2.873452(-8)
0.3	4.174528(-9)	1.786324(-8)	3.641453(-9)	2.257713(-8)
0.4	3.036036(-9)	1.299143(-8)	2.648343(-9)	1.641972(-8)
0.5	1.897518(-9)	8.119643(-9)	1.655208(-9)	1.026232(-8)
0.6	7.590100(-10)	3.247856(-9)	6.620856(-10)	4.104928(-9)
0.7	3.795042(-10)	1.623928(-9)	3.310421(-10)	2.052464(-9)
0.8	1.518015(-9)	6.495721(-9)	1.324166(-9)	8.209863(-8)
0.9	2.656530(-9)	1.136750(-8)	2.317297(-9)	1.436725(-8)
1.0	3.795042(-9)	1.623928(-8)	3.310421(-9)	2.052464(-8)
r	$E^+(0.5;r)$	$E^+(0.5;r)$	$E^+(1;r)$	$E^+(1;r)$
0.0	7.590100(-9)	3.247856(-8)	6.620856(-9)	4.104928(-8)
0.1	7.210590(-9)	3.085463(-8)	6.289804(-9)	3.899682(-8)
0.2	6.831070(-9)	2.923064(-8)	5.958752(-9)	3.694430(-8)
0.3	6.451570(-9)	2.760677(-8)	5.627722(-9)	3.489188(-8)
0.4	6.072060(-9)	2.598287(-8)	5.296667(-9)	3.283943(-8)
0.5	5.692550(-9)	2.435890(-8)	4.965625(-9)	3.078694(-8)
0.6	5.313055(-9)	2.273503(-8)	4.634590(-9)	2.873452(-8)
0.7	4.933553(-9)	2.111106(-8)	4.303548(-9)	2.668203(-8)
0.8	4.554036(-9)	1.948713(-8)	3.972496(-9)	2.462956(-8)
0.9	4.174528(-9)	1.786324(-8)	3.641453(-9)	2.257713(-8)
1.0	3.795042(-9)	1.623928(-8)	3.310421(-9)	2.052464(-8)

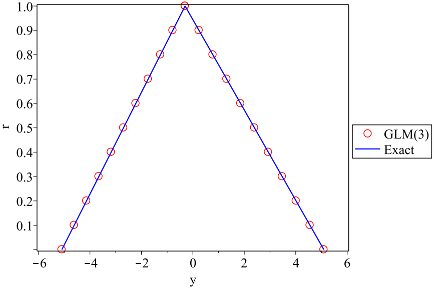


Figure 2(a)

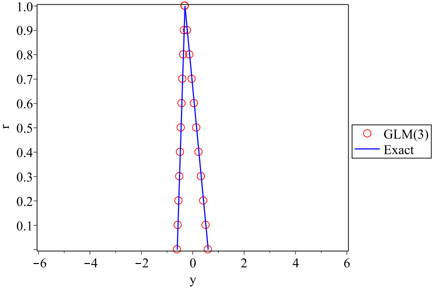


Figure 2(b)

Figure 2. (a) circle-point: GLM(3), Line: Exact with (i)-differentiability; (b) circle-point: GLM(3), Line: Exact with (ii)-differentiability

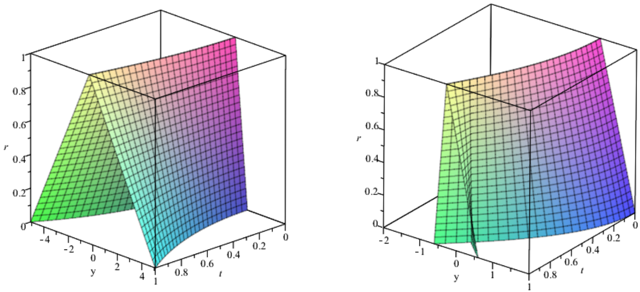


Figure 2(c) Figure 2(d)

Figure 2. (c) 3D-plot with (i)-differentiability; (b) 3D-plot with (ii)-differentiability

Conclusion

- Simpson's rule and Lagrange interpolating polynomial are explored.
- Based on these integration methods along with the generalized Hukuhara differetiability, the fuzzy version of third order GLM for solving FVIDEs are developed.
- Test results showed that GLM performed better than RK method of same order.

References

- Butcher, J.C.: General Linear Methods. Acta Numerica. 15, 157-256 (2006).
- Butcher, J.C.: General linear methods for ordinary differential equations. Mathematics and Computers in Simulation. 79, 1834–1845 (2009).
- Butcher, J.C.: Numerical Methods for Ordinary Differential Equations 2nd Edition. John Wiley and Sons (2008).
- Chartier, P., Hairer, E., Vilmart, G.: Algebraic Structures of B-series. Foundations of Computational Mathematics. 10, 407–427 (2010).
- Linz, P. 1985. Analytical and numerical methods for Volterra equations. SIAM Philadelphia.
- Linz, P.: Analytical and Numerical Methods for Volterra Equations. Philadelphia, S.I.A.M. (1985).
- Raftari, B.: Numerical solutions of the linear volterra integro-differential equations: Homotopy perturbation method and finite difference method. World Applied Sciences Journal. 9, 7–12 (2010).
- Zarebnia, M.: Sinc numerical solution for the Volterra integro-differential equation. Commun. Nonlinear Sci. Numer. Simulat. 15, 700-706 (2010).
- Filiz, A.: A fourth-order robust numerical method for integro-differential equations. Asian Journal of Fuzzy and Applied Mathematics. 1, 28–33 (2013).

THANK YOU